TSW
Test Proof Training Dec 2010 Timisoara

Anonymous

Note: the proofs should be done in natural style. Note also that we use $P[x]$ and $f[x]$ for "P applied to x" and "f applied to x", respectively (instead of the usual $P(x), f(x)$).

1. We consider the following partial proof attempt of $((A \wedge B) \Rightarrow C) \;\Rightarrow\; ((A \Rightarrow C) \vee (B \Rightarrow C))$:
   (A) We assume: $(A \wedge B) \Rightarrow C$ and we prove: $(A \Rightarrow C) \vee (B \Rightarrow C)$.
   (B) We assume: $\neg(B \Rightarrow C)$ and we prove: $(A \Rightarrow C)$.
   (C) We assume: A and we prove: C.
   (D) From $\neg(B \Rightarrow C)$ we obtain B and $\neg C$.

For each statement below, decide whether it is correct or incorrect and explain the reasons:
   (a) The proof fails because we obtain $\neg C$ instead of the current goal C.
   (b) The proof fails because we obtain a contradiction.
   (c) The proof succeeds because we obtain a contradiction.
   (d) The proof is incorrect. (Indicate the incorrect steps.)
   (e) The proof is correct but not yet finished. (Indicate how it should continue.)
   (f) The step (D) of the proof is wrong because D does not occur in the formula.
   (g) The proof is correct but the result is wrong.
   (h) We proved the negation of the initial formula, because we obtain the negation of the current goal.

2. For each symbol occuring in the following formula, specify whether it is a: logical quantifier, logical connective, predicate symbol, function symbol, variable, or constant. (Note that functions and predicates can also be constant or variable.)

$$\forall_f C[f] \Leftrightarrow \forall_{\epsilon>0}\; \exists_{\delta>0}\; \forall_{x,y}\; (|x-y| < \delta \;\Rightarrow\; |f[x] - f[y]| < \epsilon)$$

3. In the previous definition, C denotes uniform continuity, f denotes a real function of real argument, and x, y, δ and ϵ denote real numbers. Formulate and prove the statement that the sum of uniformly continuos functions is uniformly continuos. Emphasize the properties from the theory of real numbers which are necessary for this proof (definition of $f_1 + f_2$, properties of minimum and of absolute value, etc.).

4. Based on the previous definition, show that the product of uniformly continuous functions is **not** uniformly continuous, by using a counterexample (e. g. both functions are the identity).

5. In the following formulae, t stands for a tuple (i. e. list of elements). Examples of tuples are: $\langle\rangle$ (the empty list), $\langle a, b\rangle$ (a list with two elements).
The binary infix function $\smile$ concatenates two tuples. Examples:

$$\langle\rangle \smile \langle a, b, c\rangle = \langle a, b, c\rangle.$$

$$\langle a, b\rangle \smile \langle b, c\rangle = \langle a, b, b, c\rangle.$$

Consider the following definitions:

$$F[\langle\rangle] = \langle\rangle$$
$$\forall_a \forall_t \; F[\langle a\rangle \smile t] = F[t] \smile \langle a\rangle$$
$$\forall_s \; G[\; \langle\rangle, s] = s$$
$$\forall_a \forall_{t,s} \; G[\langle a\rangle \smile t, \;\; s] = G[t, \;\; \langle a\rangle \smile s,]$$

Use these equalities as rewrite rules in order to compute the expressions: $F[\langle a, b, c\rangle]$ and $G[\langle a, b, c\rangle, \; \langle\rangle]$.

6. Using the formulae above, prove:

$$\forall_t \; F[t] = G[t, \;\; \langle\rangle].$$

Hint: prove first $\forall_t \; \forall_s \; F[t] \smile s = G[t, \;\; s]$. For proving the latter, consider the predicate $P[t]$ defined as $\forall_s \; F[t] \smile s = G[t, \;\; s]$ and use the induction principle for tuples in order to prove $\forall_t \; P[t]$. (One must prove $P[\langle\rangle]$ and $\forall_a \forall_t \; (P[t] \Rightarrow P[\langle a\rangle \smile \; t])$.) Note that for proving equalities it is enough to transform both sides by using known equalities as rewrite rules – and, of course, if necessary, the appropriate properties of tuples).