Proving in Natural Style

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Contents

Chapter 1

Natural Style

1.1 Sequents as Models of Proof Situations

Example:

- Prove: $(A \wedge (A \Rightarrow B)) \Rightarrow B$
- "deduction rule"
  - Prove: B
  - Assume: $A \wedge (A \Rightarrow B)$
    - this is decomposed into parts ...
      - Assume: A
      - Assume: $A \Rightarrow B$
    - ... by "modus ponens":
      - Know: B

the *formal steps for this proof*:

$$
\begin{array}{rl}
\{\} \vdash (A \wedge (A \Rightarrow B) \Rightarrow B & \quad \text{("Proof situation")} \\
\downarrow_1 & \\
\{A \wedge (A \Rightarrow B)\} \vdash B & \\
\downarrow_2 & \\
\{A, A \Rightarrow B\} \vdash B & \\
\downarrow_3 & \\
\{A, A \Rightarrow B, \underline{B}\} \vdash \underline{B} & \\
\downarrow_4 & \\
\mathbb{T} &
\end{array}
$$

General: *"rewriting rules"*

Note: $\Phi \vdash \Psi$ is called a *"sequent"*

$$\text{R1 (rule 1)} \quad = \quad \begin{cases} \Phi & \vdash \quad \psi_1 \Rightarrow \psi_2 \\ & \downarrow \\ \Phi \cup \{\psi_1\} & \vdash \quad \psi_2 \end{cases}$$

$$\text{R2 (rule 2)} \quad = \quad \begin{cases} \Phi \cup \{\varphi_1 \wedge \varphi_2\} & \vdash \quad \Psi \\ & \downarrow \\ \Phi \cup \{\varphi_1, \varphi_2\} & \vdash \quad \Psi \end{cases}$$

$$\begin{matrix}\text{R3 (rule 3)} \\ \text{modus ponens}\end{matrix} \quad = \quad \begin{cases} \Phi \cup \{\varphi_1, \varphi_1 \Rightarrow \varphi_2\} & \vdash \quad \Psi \\ & \downarrow \\ \Phi \cup \{\varphi_1, \varphi_1 \Rightarrow \varphi_2, \varphi_2\} & \vdash \quad \Psi \end{cases}$$

$$\text{R4 (rule 4)} \quad = \quad \begin{cases} \Phi & \vdash \quad \psi \\ & \downarrow \quad \text{if } \psi \in \Phi \\ & \mathbb{T} \end{cases}$$

Proofs are written like this (from below to above, but without the arrow):

$$\uparrow \qquad \text{("premises")} \quad \cfrac{\cfrac{\cfrac{\cfrac{\mathbb{T}}{A, A \Rightarrow B, B \vdash B}}{A, A \Rightarrow B \vdash B}}{A \wedge (A \Rightarrow B) \vdash B}}{\vdash (A \wedge (A \Rightarrow B)) \Rightarrow B}$$

Inference rules are written like that:

$$\frac{\Phi, \psi_1 \vdash \psi_2}{\Phi \vdash \psi_1 \Rightarrow \psi_2} \qquad \text{(R1)}$$

$$\frac{\Phi, \varphi_1, \varphi_2 \vdash \Psi}{\Phi, \varphi_1 \wedge \varphi_2 \vdash \Psi} \qquad \text{(R2)}$$

$$\frac{\Phi, \varphi_1, \varphi_1 \Rightarrow \varphi_2, \varphi_2 \vdash \Psi}{\Phi, \varphi_1, \varphi_1 \Rightarrow \varphi_2 \vdash \Psi} \qquad \text{(R3)}$$

... and read like that:

$$\frac{\text{If we are able to prove this}}{\text{then we can be sure that this holds}}$$

Keep in mind:

- We want to develop rules that help us to construct proofs
- To actually construct a proof one must combine these rules

$$\Phi, \varphi \vdash \varphi \text{ "is axiom"}$$

Example:

Prove: $\underbrace{B \vee C, B \Rightarrow A, C \Rightarrow D}_{\text{Assume}} \;\vdash\; \underbrace{A \vee D}_{\text{Prove}}$

"Prove by cases"

↙ ↘

Assume B	Assume C
by $B \Rightarrow A$ (modus ponens) ↓	↓ by $C \Rightarrow D$ (modus ponens)
Know A	Know D
Know $A \vee D$, Q.E.D.	Know $A \vee D$, Q.E.D.

Figure 1.1: Informal Proof Tree

Sequent

$$\Phi, \varphi \quad \vdash \quad \varphi \vee \psi$$

considered as conjunction   considered as disjunction

$$\Phi, \varphi \quad \vdash \quad \Psi, \varphi \quad \text{"Axiom"}$$

So we consider working with two sets:

$$\text{"Sequent":} \quad \underset{\text{conjunction}}{\Phi} \quad \vdash \quad \underset{\text{disjunction}}{\Psi}$$

Definition: A *"Sequent"* is a pair of two formulae.
The set on the LHS is a conjunction
The set on the RHS is a disjunction

The sequent $\Phi \vdash \Psi$ holds iff $\bigwedge \Phi \Rightarrow \bigvee \Psi$

Note: $\bigwedge$ (analogically $\bigvee$) means the AND- (analogically OR-)Operation over all set elements

$$\varphi_1, \ldots, \varphi_n \vdash \psi_1, \ldots, \psi_m \text{ holds}$$
$$\text{iff } (\varphi_1 \wedge \cdots \wedge \varphi_n) \Rightarrow (\psi_1 \vee \cdots \vee \psi_m) \text{ is valid}$$

Proof Tree

$$\dfrac{\dfrac{\dfrac{B, A, B \Rightarrow A, C \Rightarrow D \vdash A, D \text{ ("axiom")}}{B, A, B \Rightarrow A, C \Rightarrow D \vdash A \vee D}}{B, B \Rightarrow A, C \Rightarrow D \vdash A \vee D} \qquad \dfrac{\dfrac{C, D, B \Rightarrow A, C \Rightarrow D \vdash A, D \text{ ("axiom")}}{C, D, B \Rightarrow A, C \Rightarrow D \vdash A \vee D}}{C, B \Rightarrow A, C \Rightarrow D \vdash A \vee D}}{B \vee C, B \Rightarrow A, C \Rightarrow D \vdash A \vee D}$$

$$\frac{\Phi, \varphi_1 \vdash \Psi \qquad \Phi, \varphi_2 \vdash \Psi}{\Phi, \varphi_1 \vee \varphi_2 \vdash \Psi}$$

Figure 1.2: Rule for cases (“Fallunterscheidung”) ($\vee \vdash$)

$$\frac{\Phi \vdash \Psi, \psi_1, \psi_2}{\Phi \vdash \Psi, \psi_1 \vee \psi_2}$$

Figure 1.3: R2 ($\vdash \vee$)

As already mentioned above (see 1.1 on the preceding page), for a sequent $\Phi \vdash \Psi$:

$$\varphi_1, \ldots, \varphi_n \vdash \psi_1, \ldots, \psi_m \text{ holds}$$
$$\text{iff } (\varphi_1 \wedge \cdots \wedge \varphi_n) \Rightarrow (\psi_1 \vee \cdots \vee \psi_m) \text{ is valid}$$

so we can say by applying rule $(A \vdash B) \equiv (\neg A \vee B)$

$$\varphi_1, \ldots, \varphi_n \vdash \psi_1, \ldots, \psi_m \text{ holds}$$
$$\text{iff } (\neg\varphi_1 \vee \cdots \vee \neg\varphi_n) \vee (\psi_1 \vee \cdots \vee \psi_m) \text{ is valid}$$

So we see that

$$\begin{array}{ccc} \bigwedge \Phi & \Rightarrow & \bigvee \Psi \\ & \equiv & \\ \neg \bigwedge \Phi & \vee & \bigvee \Psi \end{array}$$

This also enables us to move (negated) clauses from Φ to Ψ (and vice versa) because:

$$\begin{array}{rcl} (\neg\varphi_1 \vee \cdots \vee \neg\varphi_n \vee \neg\psi_1) & \vee & (\psi_2) \\ & \downarrow & (\text{moving } \neg\psi_1) \\ (\neg\varphi_1 \vee \cdots \vee \neg\varphi_n) & \vee & (\neg\psi_1 \vee \psi_2) \end{array}$$

Example

Prove $((A \vee B) \Rightarrow C) \Rightarrow ((A \Rightarrow C) \wedge (B \Rightarrow C))$
Assume $(A \vee B) \Rightarrow C$
Prove $(A \Rightarrow C) \wedge (B \Rightarrow C)$

$\swarrow$ $\searrow$

Prove $A \Rightarrow C$
Assume A, Know: $A \vee B$ $\overrightarrow{\text{by MP}}$ C
Prove C

Prove $B \Rightarrow C$
Assume B, Know: $A \vee B$ $\overrightarrow{\text{by MP}}$ C
Prove C

Note: MP means “modus ponens” here

$$\frac{\Phi \vdash \Psi, \psi_1 \qquad \Phi \vdash \Psi, \psi_2}{\Phi \vdash \Psi, \psi_1 \wedge \psi_2}$$

Figure 1.4: Rule ($\vdash \wedge$)

Proof tree:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\neg B, A \vdash A, C \text{ (“axiom”)}}{\neg A, \neg B, A \vdash C}}{\neg A \wedge \neg B, A \vdash C} \quad C, A \vdash C \text{ (“axiom”)}}{\neg(A \vee B) \vee C, A \vdash C}}{(A \vee B) \Rightarrow C, A \vdash C}}{(A \vee B) \Rightarrow C \vdash A \Rightarrow C} \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\neg A, B \vdash B, C \text{ (“axiom”)}}{\neg A, \neg B, B \vdash C}}{\neg A \wedge \neg B, B \vdash C} \quad C, B \vdash C \text{ (“axiom”)}}{\neg(A \vee B) \vee C, B \vdash C}}{(A \vee B) \Rightarrow C, B \vdash C}}{(A \vee B) \Rightarrow C \vdash B \Rightarrow C}}{((A \vee B) \Rightarrow C) \vdash ((A \Rightarrow C) \wedge (B \Rightarrow C))}$$

Summary

- *”sequent”:* $\Phi \vdash \Psi$
- *“sequent holds”:* $\bigwedge \Phi \Rightarrow \bigvee \Psi$
- *“inference rule”:* $\frac{S_1 \ldots S_n}{S}$ iff (if $S_1, \ldots, S_n$ hold, then S holds)
  - Set of inference rules: *“calculus”*
- *proof (tree):* tree
  - a *tree* is a graph with properties
    - root (is ancestor of all nodes)
    - leaves (nodes which have no successors)
    - some nodes have successors
  - a tree *“is a proof of S”* iff
    1. S is the root of the tree
    2. Whenever $S_1, \ldots, S_n$ are successors of a node S they are an instance of an inference rule.
    3. The leaves are the axioms

1.2 The Small Calculus

We construct a calculus (set of inference rules) for formulae containing only: $\neg, \wedge, \vee$

	premises		conclusions	
$\neg$	$\dfrac{\Phi \vdash \Psi, \varphi}{\Phi, \neg\varphi \vdash \Psi}$	$(\neg \vdash)$	$\dfrac{\Phi, \psi \vdash \Psi}{\Phi \vdash \Psi, \neg\psi}$	$(\vdash \neg)$
$\wedge$	$\dfrac{\Phi, \varphi_1, \varphi_2 \vdash \Psi}{\Phi, \varphi_1 \wedge \varphi_2 \vdash \Psi}$	$(\wedge \vdash)$	$\dfrac{\Phi \vdash \Psi, \psi_1 \quad \Phi \vdash \Psi, \psi_2}{\Phi \vdash \Psi, \psi_1 \wedge \psi_2}$	$(\vdash \wedge)$
$\vee$	$\dfrac{\Phi, \varphi_1 \vdash \Psi \quad \Phi, \varphi_2 \vdash \Psi}{\Phi, \varphi_1 \vee \varphi_2 \vdash \Psi}$	$(\vee \vdash)$	$\dfrac{\Phi \vdash \Psi, \psi_1, \psi_2}{\Phi \vdash \Psi, \psi_1 \vee \psi_2}$	$(\vdash \vee)$

Axioms:

- $\Phi, \varphi \vdash \Psi, \varphi$ or alternatively:
- $\Phi \vdash \Psi$ is axiom iff $\Phi \cap \Psi \neq \emptyset$

Desired properties

- *“correct”* iff (if a sequent has a proof, then the sequent holds)

- *"complete"* iff (if a sequent holds, then it has a proof)

This can also be expressed in traditional logic:

$$\Phi \vdash \Psi \quad \text{iff} \quad \Phi \models \Psi$$

Exercise: Prove informally: $\Phi \vdash \Psi$ holds iff $\Phi \models \Psi$

Remark: $\Phi \models \Psi$ can be read as "is a semantical logical consequence"

Correctness can be proven by proving the correctness of each individual rule. Moreover on can prove that each of these rules is "reversible": if S holds, then each of S_1, S_2 hold.

Completeness holds by the following argument: For every sequent one can construct a tree by applying the inference rules, whose leaves contain only "atomic" sequents (without logical connectives – the premises and the conclusions are all atomic). This is because there is a rule for each logical connective in the premises and in the conclusions, and each rule eliminates the logical connective.
Because of the reversibility of the rules, by induction, if the root is a valid sequent, then all leaves are also valid. However an atomic sequent can be valid only if it is an axiom (see below).

Axioms. If a sequent $\Phi \vdash \Psi$ is atomic (Φ and *Psi* contain only atoms), then:
The sequent holds if and only if $\Phi \cap \Psi \neq \emptyset$.

Proof
Right to left: Let $A = \Phi \cap \Psi$. When we write the normal form of the formula corresponding to the sequent as a disjunction, then both $\neg A$ (from Φ) and A (from Ψ) will occur in it, thus it reduces to $\mathbb{T}$.
Left to right: By contradiction. If the $\Phi \cap \Psi = \emptyset$, then we can construct the interpretation in which all elements of Φ are assigned $\mathbb{T}$ and all elements of Ψ are assigned $\mathbb{F}$, and this interpretation falsifies the formula corresponding to the sequent.

Remark: A tree will be constructed by this calculus even if the sequent is not valid. In this case, some of the leaves will not be axioms, and by the construction of the interpretation given in the proof of the previous lemma, we obtain a counterexample on each such leave. The counterexamples may be important in practice, because they exhibit, for instance, bugs in the design of the system (hardware or software) which we want to verify by applying the prover.

Empty sets in sequents

In a sequent $\Phi \vdash \Psi$, any of the sets Φ and *Psi* may be empty. The meaning of $\bigvee \Psi$ is $\mathbb{F}$, and the meaning of $\bigwedge \Psi$ is $\mathbb{T}$, as it is explained below.

$\bigvee \Psi$	$(\exists \psi : \psi \in \Psi) \langle \psi \rangle_I = \mathbb{T}$	$(\exists \psi) : \psi \in \Psi \wedge \langle \psi \rangle_I = \mathbb{T}$
$\bigwedge \Psi$	$(\forall \psi : \psi \in \Psi) \langle \psi \rangle_I = \mathbb{T}$	$(\forall \psi) : \psi \in \Psi \Rightarrow \langle \psi \rangle_I = \mathbb{T}$

Analogy:

$\exists x \in \mathbb{R} : x \in \mathbb{Q}$	$(\exists x) x \in \mathbb{R} \wedge x \in \mathbb{Q}$	incorrect: ~~$(\exists x) x \in \mathbb{R} \Rightarrow x \in \mathbb{Q}$~~ ~~$(\exists x) x \notin \mathbb{R} \vee x \in \mathbb{Q}$~~
$\forall x \in \mathbb{Q} : x \in \mathbb{R}$	$(\forall x) x \in \mathbb{Q} \Rightarrow x \in \mathbb{R}$	incorrect: ~~$(\forall x) x \in \mathbb{Q} \wedge x \in \mathbb{R}$~~

$$\langle \bigvee \Phi \rangle_I = \mathbb{T} \quad \text{iff} \quad \exists \varphi \in \Phi : \langle \varphi \rangle_I = \mathbb{T} \qquad (\exists \varphi) \varphi \in \Phi \wedge \langle \varphi \rangle_I = \mathbb{T}$$

$$\langle \bigvee \{\} \rangle_I = \mathbb{F} \qquad \left. (\exists \varphi) \underbrace{\varphi \in \{\}}_{\mathbb{F}} \wedge \ldots \right\} \mathbb{F}$$

$$\langle \bigwedge \Phi \rangle_I = \mathbb{T} \quad \text{iff} \quad \forall \varphi \in \Phi : \langle \varphi \rangle_I = \mathbb{T} \qquad (\forall \varphi) \varphi \in \Phi \Rightarrow \langle \varphi \rangle_I = \mathbb{T}$$

$$\langle \bigwedge \{\} \rangle_I = \mathbb{T} \qquad \left. (\forall \varphi) \underbrace{\varphi \in \{\}}_{\mathbb{F}} \Rightarrow \ldots \right\} \mathbb{T}$$

Sequent $\Phi \vdash \Psi$

$\swarrow \qquad \searrow$

if left side is empty: $\{\} \vdash \Psi$	if right side is empty: $\Phi \vdash \{\}$
also written: $\vdash \Psi$	also written: $\Phi \vdash$
(this is like having no assumptions)	

$$(\Phi \vdash \Psi) \text{ holds} \quad \text{iff} \quad (\bigwedge \Phi \Rightarrow \bigvee \Psi) \text{ is valid}$$

$$(\{\varphi_1, \ldots, \varphi_n\} \vdash \{\psi_1, \ldots, \psi_m\}) \quad \text{iff} \quad \underbrace{((\varphi_1 \wedge \cdots \wedge \varphi_n) \Rightarrow (\psi_1 \vee \cdots \wedge \psi_m))}_{} \text{ is valid}$$

$$\underbrace{(\neg(\varphi_1 \wedge \cdots \wedge \varphi_n) \vee (\psi_1 \vee \cdots \wedge \psi_m))}_{}$$

$$(\neg \varphi_1 \vee \cdots \vee \neg \varphi_n \vee \psi_1 \vee \cdots \wedge \psi_m)$$

1.3 Predicate Logic

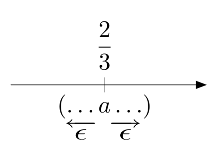
Example:

A sequence f is convergent: $\underbrace{\exists_{a \in \mathbb{R}} \forall_{\substack{\epsilon \in \mathbb{R} \\ \epsilon > 0}} \exists_{n \in \mathbb{N}} \forall_{\substack{p \in \mathbb{N} \\ p > n}} |f(p) - a| < \epsilon}_{\varphi}$

$f : \mathbb{N} \to \mathbb{R}$

$f(n) = \frac{2n+3}{3n+1}$

n	$f(n)$
0	3/1
1	5/4
2	7/7
3	9/10
$\vdots$	$\vdots$



"in our syntax":

$$\exists_a (a \in \mathbb{R}) \wedge \forall_\epsilon (\epsilon \in \mathbb{R} \wedge \epsilon > 0) \Rightarrow \exists_n (n \in \mathbb{N}) \wedge \forall_p (p \in \mathbb{N} \wedge p > n) \Rightarrow |f(p) - a| < \epsilon$$

Note that $|f(p) - a| < \epsilon$ could also be formulated as $Less(Abs(Minus(f(p), a)), \epsilon)$

$$\forall_f isSequence(f) \Rightarrow (isConvergent(f) \Leftrightarrow \varphi)$$

Note that *isConvergent* is a predicate over a function, so this is "second order" predicate logic, where one can have predicates and functions applied to first order formulae. The general term is "higher order" predicate logic.

$$\left.\begin{array}{l}\text{If } f, g \text{ convergent, then } f \oplus g \text{ is also convergent:} \\ \forall_{f,g} \forall_{n \in \mathbb{N}} \quad \underbrace{(f \oplus g)(n)}_{Plus(f,g)(n) \text{ "currying"}} = f(n) + g(n)\end{array}\right\} \text{higher order}$$

$$\forall_{\substack{f,g \\ IsSequence(f) \wedge \\ IsSequence(g)}} (IsConvergent(f) \wedge IsConvergent(g)) \Rightarrow IsConvergent(f \oplus g)$$

Note: f, g are arbitary but fixed here

$f, g :\quad IsSequence(f),$ $IsSequence(g)$

$$\exists_{a \in \mathbb{R}} \forall_{\substack{\epsilon \in \mathbb{R} \\ \epsilon > 0}} \exists_{n \in \mathbb{N}} \forall_{\substack{p \in \mathbb{N} \\ p > n}} |f(p) - a| < \epsilon$$

Assume: $\begin{cases} IsConvergent(f) \nearrow \\ IsConvergent(g) \searrow \end{cases}$

$$\exists_{a \in \mathbb{R}} \forall_{\substack{\epsilon \in \mathbb{R} \\ \epsilon > 0}} \exists_{n \in \mathbb{N}} \forall_{\substack{p \in \mathbb{N} \\ p > n}} |g(p) - a| < \epsilon$$

Prove: $IsConvergent(f \oplus g)$

$$\exists_{a \in \mathbb{R}} \forall_{\substack{\epsilon \in \mathbb{R} \\ \epsilon > 0}} \exists_{n \in \mathbb{N}} \forall_{\substack{p \in \mathbb{N} \\ p > n}} \left| \underbrace{(f \oplus g)(p)}_{\substack{\text{replace with:} \\ f(p) + g(p)}} - a \right| < \epsilon$$

$$\text{Used for proof} \begin{cases} \forall_x IsSequence(f) \Rightarrow (IsConvergent(f) \Leftrightarrow \varphi) \\ \xrightarrow{\text{instantiation}} \quad IsSequence(g) \Rightarrow \underbrace{\left(isConvergent(g) \Leftrightarrow \varphi_{f \leftarrow g}\right)}_{\forall_x \varphi \rightsquigarrow \varphi_{x \leftarrow t}} \\ \frac{\cdots \vdash \varphi}{\varphi \Leftrightarrow \psi \vdash \Psi} \end{cases}$$

A_f : "take" $a_1 \in \mathbb{R}$:	$\forall_{\substack{\epsilon \in \mathbb{R} \\ \epsilon > 0}} \ldots f \cdots < \frac{\epsilon_0}{2}$
A_g : "take" $a_2 \in \mathbb{R}$:	$\forall_{\substack{\epsilon \in \mathbb{R} \\ \epsilon > 0}} \ldots g \cdots < \frac{\epsilon_0}{2}$
G : "use" $a \leftarrow a_1 + a_2$	$\forall_{\substack{\epsilon \in \mathbb{R} \\ \epsilon > 0}} \exists_{n \in \mathbb{N}} \lvert (f(p) + g(p)) - (a_1 + a_2) \rvert < \epsilon_0$ $\epsilon_0 \in \mathbb{R}$ $a_1 + a_2 \in \mathbb{R}$ $\quad \epsilon_0 > 0$ assumptions

Exercise: finish this proof

inference rules (assumptions)	inference rules (goals)
$\forall_x\varphi \vdash\rightsquigarrow \varphi_{x\leftarrow t} \vdash$	$\vdash \forall_x\varphi \rightsquigarrow\vdash \varphi_{x\leftarrow a}$ (a "is new") ($a \mathring{\notin} \varphi$)
$\exists_x\varphi \vdash\rightsquigarrow \varphi_{x\leftarrow a} \vdash$ (a "is new") ($a \mathring{\notin} \varphi$)	$\vdash \exists_x\varphi \rightsquigarrow\vdash \varphi_{x\leftarrow t}$

The difficulty of finding appropriate terms.

Note that the rules are effective only when x occurs in φ. $\varphi_{x\rightarrow t}$ denotes the formula which is obtained by replacing in φ each occurence of x by t. t stands for an arbitrary ground term.

The inference rules presented above and the inference rules for propositional logic presented in the previous section are easy to automate, except for the two ones which involve the substitution with a term t, because it is not obvious how to find this term.

In clausal proving (presented in the previous chapters), the Herbrand theorem suggests a proof procedure in which these terms are found by “trial and error”: one just enumerates all the ground terms (this is the Herbrand universe!) until the righ ones are found. Resolution provides a more efficient procedure, because the terms found by unification have higher chance to be useful for the proof.

In natural style proving one can use similar strategies, and finding a very efficient one is still a matter of scientific research.

Correctness.

Correctness of the rules $\exists \vdash$

1.4 Sequents

inference rules (assumptions)	inference rules (goals)
$\dfrac{\Phi, \varphi_{x\leftarrow a} \vdash \Psi}{\Phi, \exists_x\varphi \vdash \Psi}$ $(\exists \vdash)$ where $a \mathring{\notin} \varphi, \Phi, \Psi$	$\dfrac{\Phi \vdash \Psi, \psi_{x\leftarrow t}}{\Phi \vdash \Psi, \exists_x\psi}$ $(\vdash \exists)$
$\dfrac{\Phi, \varphi_{x\leftarrow t} \vdash \Psi}{\Phi, \forall_x\varphi \vdash \Psi}$ $(\forall \vdash)$	$\dfrac{\Phi \vdash \Psi, \psi_{x\leftarrow a}}{\Phi \vdash \Psi, \forall_x\psi}$ $(\vdash \forall)$ where $a \mathring{\notin} \psi, \Phi, \Psi$

Note: first we will study the proof system (resolution) and we will infer as a consequence the proof for correctness and completeness

1.4.1 Q1: Difference Correctness / Completeness

With the sequent calculus:

- *Correctness:* if $\left(\varphi \begin{array}{c} \text{(obtain a proof tree)} \\ \downarrow \\ \text{syntactically} \\ \downarrow \\ \text{proved} \end{array} \right) \xrightarrow{\text{then}} (\varphi \text{ is valid})$

- *Completeness:* Whenever (φ is valid) $\overleftarrow{\text{then}}$ (you are able to prove φ)

And with resolution:

- *Correct:* Whenever you can obtain □, you know that it is unsatisfiable
- *Complete:* Whenever it is unsatisfiable, you will obtain □

Sequent calculus:

$\frac{S_1 S_2}{S}$ Correctness/Completeness: Because both directions (↑↓) work for each rule

1.4.2 Q2: Universal quantifiers

inference rules (assumptions)	inference rules (goals)
$\dfrac{\Phi, \varphi_{x\leftarrow t} \vdash \Psi}{\Phi, \forall_x \varphi \vdash \Psi}$ $(\forall \vdash)$	$\dfrac{\Phi \vdash \Psi, \psi_{x\leftarrow a}}{\Phi \vdash \Psi, \forall_x \psi}$ $(\vdash \forall)$ where $a \mathring{\notin} \psi, \Phi, \Psi$

For example (see example with convergence):

$$\begin{array}{lcccl} \text{replace } \epsilon & \ldots, \forall_\epsilon \ldots \text{ if } \ldots| < \epsilon & \vdash & \forall_\epsilon \ldots \text{ if } f \oplus g \ldots| < \epsilon & \leftarrow \text{ in order to prove it for all it is enough} \ldots \\ \text{with a new term} \downarrow & \text{if } \ldots| < \frac{\epsilon_0}{2} & \vdash & \ldots \text{ if } f \oplus g \ldots| < \epsilon_0 & \leftarrow \ldots \text{ to prove it for one constant which somehow looks strange} \end{array}$$

The clou of the proof is how define this term t.

1.4.3 Q3: Predicate logic proof tree

Example showing the essence:

$$\dfrac{\dfrac{\text{modus ponens}}{P(a) \Rightarrow P(f(a)), P(f(a)) \Rightarrow P(f(f(a))), P(a) \vdash P(f(f(a)))}}{\forall_x P(x) \Rightarrow P(f(x)), P(a) \vdash P(f(f(a)))}$$

This corresponds to $A \Rightarrow B, B \Rightarrow C, A \vdash C$

or

$$\dfrac{\ldots\ldots\ldots\ldots\ldots\ldots \vdash P(f(f(a)))}{\forall_x P(x) \Rightarrow P(f(x)), P(a) \vdash \exists_x P(f(f(x)))} \quad \text{(replace with } a) \quad \boxed{\leftrightarrow \quad \dfrac{\vdash \psi_{x\leftarrow t}}{\vdash \exists_x \psi} \text{ (replace with term)}}$$

or

$$\dfrac{\forall_x P(x) \Rightarrow P(f(x)), P(a) \vdash \forall_x P(f(f(x)))}{\forall_x P(x) \Rightarrow P(f(x)), \forall_x P(x) \vdash \forall_x P(f(f(b)))} \quad P(f(f(b)))$$